A CONNECTION BETWEEN DECOMPOSABLE ULTRAFILTERS AND POSSIBLE COFINALITIES. II

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Abstract. We use Shelah's theory of possible cofinalities in order to solve a problem about ultrafilters.

Theorem 1. *Suppose that λ is a singular cardinal, $\lambda' < \lambda$, and the ultrafilter D is κ-decomposable for all regular cardinals κ with $\lambda' < \kappa < \lambda$. Then D is either λ-decomposable, or λ^+-decomposable.*

We give applications to topological spaces and to abstract logics (Corollaries 7, 8 and Theorem 9).

If F is a family of subsets of some set I, and λ is an infinite cardinal, a λ-*decomposition* for F is a function $f : I \to \lambda$ such that whenever $X \subseteq \lambda$ and $|X| < \lambda$ then $\{i \in I | f(i) \in X\} \notin F$. The family F is λ-*decomposable* if and only if there is a λ-decomposition for F. If D is an ultrafilter (that is, a maximal proper filter) let us define the *decomposability spectrum* K_D of D by $K_D = \{\lambda \geq \omega | D \text{ is } \lambda\text{-decomposable}\}$.

The question of the possible values the spectrum K_D may take is particularly intriguing. Even the old problem from [Si] of characterizing those μ for which there is an ultrafilter D such that $K_D = \{\omega, \mu\}$ is not yet completely solved [Shr, p. 1007].

The case when K_D is infinite is even more involved. [P] studied the situation in which λ is limit and $K_D \cap \lambda$ is unbounded in λ; he found some assumptions which imply that $\lambda \in K_D$. This is not always the case; if μ is strongly compact and $\operatorname{cf} \lambda < \mu < \lambda$ then there is an ultrafilter D such that $K_D \cap \lambda$ is unbounded in λ, and D is not λ-decomposable. If we are in the above situation, we have that necessarily D is λ^+-decomposable (by [So, Lemma 3] and the proof of [P, Proposition 2]).

The above examples suggest the problem whether $K_D \cap \lambda$ unbounded in λ implies that either $\lambda \in K_D$ or $\lambda^+ \in K_D$. In general, the problem

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is still open; here we solve it affirmatively in the particular case when there is $\lambda' < \lambda$ such that K_D contains all regular cardinals in the interval $[\lambda', \lambda)$; moreover, when $\mathrm{cf}\,\lambda > \omega$ it is sufficient to assume that $\{\kappa < \lambda | \kappa^+ \in K_D \cap \lambda\}$ is stationary in λ.

We briefly review some known results on K_D. If κ is regular and $\kappa^+ \in K_D$ then $\kappa \in K_D$; and if $\kappa \in K_D$ is singular, then $\mathrm{cf}\,\kappa \in K_D$. Results from [D] imply that if there is no inner model with a measurable cardinal then K_D is always an interval with minimum ω. On the other hand, it is trivial that $K_D = \{\mu\}$ if and only if μ is either ω or a measurable cardinal. Further comments and constraints on K_D are given in [L4, L5]. Apparently, the problem of determining which sets of cardinals can be represented as $K_F = \{\lambda \geq \omega | F \text{ is } \lambda\text{-decomposable}\}$ for a filter F has not been studied.

If $(\lambda_j)_{j \in J}$ are regular cardinals, the *cofinality* $\mathrm{cf} \prod_{j \in J} \lambda_j$ of the product $\prod_{j \in J} \lambda_j$ is the smallest cardinality of a set $G \subseteq \prod_{j \in J} \lambda_j$ having the property that for every $f \in \prod_{j \in J} \lambda_j$ there is $g \in G$ such that $f(j) \leq g(j)$ for all $j \in J$.

We shall state our results in a quite general form, involving arbitrary filters, rather than ultrafilters. In what follows, the reader interested in ultrafilters only can always assume that F is an ultrafilter.

Proposition 2. *If $(\lambda_j)_{j \in J}$ are infinite regular cardinals, $\mu = \mathrm{cf} \prod_{j \in J} \lambda_j$ and the filter F is λ_j-decomposable for all $j \in J$, then F is μ'-decomposable for some μ' with $\sup_{j \in J} \lambda_j \leq \mu' \leq \mu$.*

Proof. Let F be over I, and let $(g_\alpha)_{\alpha \in \mu}$ witness $\mu = \mathrm{cf} \prod_{j \in J} \lambda_j$. For every $j \in J$ let $f(j, -) : I \to \lambda_j$ be a λ_j decomposition for F. For any fixed $i \in I$, $f(-, i) \in \prod_{j \in J} \lambda_j$, thus there is $\alpha(i) \in \mu$ such that $f(j, i) \leq g_{\alpha(i)}(j)$ for all $j \in J$.

Let X be a subset of μ with minimal cardinality with respect to the property that $Y = \{i \in I | \alpha(i) \in X\} \in F$. Let $\mu' = |X|$. Thus, whenever $X' \subseteq \mu$ and $|X'| < \mu'$, we have $Y' = \{i \in I | \alpha(i) \in X'\} \notin F$. Define $h(i) = \alpha(i)$ for $i \in Y$, and $h(i) = 0$ for $i \notin Y$. Thus, $h : I \to X \cup \{0\}$.

If $|X'| < \mu'$ then $\{i \in I | h(i) \in X'\} \subseteq Y' \cup (I \setminus Y) \notin F$ (otherwise, since F is a filter, $Y' \supseteq Y \cap Y' = Y \cap (Y' \cup (I \setminus Y)) \in F$, contradiction). This shows that, modulo a bijection from $X \cup \{0\}$ onto μ', h is a μ'-decomposition for F. Trivially, $\mu' \leq \mu$.

Hence, it remains to show that $\sup_{j \in J} \lambda_j \leq \mu'$. Suppose to the contrary that $\mu' < \lambda_{\bar{j}}$ for some $\bar{j} \in J$. Then $|\{g_{\alpha(i)}(\bar{j}) | i \in Y\}| \leq |\{\alpha(i) | \alpha(i) \in X\}| \leq |X| = \mu' < \lambda_{\bar{j}}$. Since $\lambda_{\bar{j}}$ is regular, we have that $\beta = \sup_{i \in Y} g_{\alpha(i)}(\bar{j}) < \lambda_{\bar{j}}$. Hence, if $i \in Y$, then $f(\bar{j}, i) \leq g_{\alpha(i)}(\bar{j}) \leq$

$\beta < \lambda_{\bar{j}}$. Thus, $|[0,\beta]| < \lambda_{\bar{j}}$, but $\{i \in I | f(\bar{j}, i) \in [0,\beta]\} \supseteq Y \in F$, and this contradicts the assumption that $f(\bar{j}, -)$ is a $\lambda_{\bar{j}}$ decomposition for F. □

Proposition 2 has not the most general form: we have results dealing with the cofinality μ of reduced products $\operatorname{cf} \prod_E \lambda_j$, where E a filter on J. We shall not need this more general version here.

Recall that an ultrafilter D is (μ, λ)-regular if and only if there is a family of λ members of D such that the intersection of any μ members of the family is empty. We list below the properties of decomposability and regularity we shall need. Much more is known: see [L2, L5] and references there.

Properties 3. *(a) Every λ-decomposable ultrafilter is* cf *λ-decomposable.*

(b) Every cf *λ-decomposable ultrafilter is (λ, λ)-regular.*

(c) If $\mu' \geq \mu$ and $\lambda' \leq \lambda$ then every (μ, λ)-regular ultrafilter is (μ', λ')-regular.

(d) [CC, Theorem 1] [KP, Theorem 2.1] *If λ is singular, D is a λ^+-decomposable ultrafilter, and D is not* cf *λ-decomposable then D is (λ', λ^+)-regular for some $\lambda' < \lambda$.*

(e) [Ka, Corollary 2.4] *If λ is singular then every λ^+-decomposable ultrafilter is (λ, λ^+)-regular.*

(f) [L1, Corollary 1.4] *If λ is singular then every (λ, λ)-regular ultrafilter is either* cf *λ-decomposable or (λ', λ)-regular for some $\lambda' < \lambda$.*

(g) If λ is regular then an ultrafilter is λ-decomposable if and only if it is (λ, λ)-regular.

Theorem 4. *Suppose that λ is a singular cardinal, F is a filter, and either*

(a) there is $\lambda' < \lambda$ such that F is κ-decomposable for all regular cardinals κ with $\lambda' < \kappa < \lambda$, or

(b) cf *$\lambda > \omega$ and $S = \{\kappa < \lambda | F$ is κ^+-decomposable$\}$ is stationary in λ.*

Then F is either λ-decomposable, or λ^+-decomposable.

If $F = D$ is an ultrafilter, then D is (λ, λ)-regular. Moreover, D is either (i) λ-decomposable, or (ii) (λ', λ^+)-regular for some $\lambda' < \lambda$, or (iii) cf *λ-decomposable and (λ, λ^+)-regular.*

Proof. Recall from [She] that if $\mathfrak{a}$ is a set of regular cardinals, then $\operatorname{pcf} \mathfrak{a}$ is the set of regular cardinals which can be obtained as $\operatorname{cf} \prod_E \mathfrak{a}$, for some ultrafilter E on $\mathfrak{a}$. If $\operatorname{cf} \lambda = \nu > \omega$ then by [She, II, Claim 2.1] there is a sequence $(\lambda_\alpha)_{\alpha \in \nu}$ closed and unbounded in λ and such that, letting $\mathfrak{a} = \{\lambda^+_\alpha | \alpha \in \nu\}$, we have $\lambda^+ = \max \operatorname{pcf} \mathfrak{a}$. If $\operatorname{cf} \lambda = \omega$

then we have $\lambda^+ = \max \mathrm{pcf}\, \mathfrak{a}$ for some countable $\mathfrak{a}$ unbounded in λ as a consequence of [She, II, Theorem 1.5] (since $\mathfrak{a}$ is countable, any ultrafilter over $\mathfrak{a}$ is either principal, or extends the dual of the ideal of bounded subsets of $\mathfrak{a}$).

Letting $\mathfrak{b} = \mathfrak{a} \cap [\lambda', \lambda)$ in case (a), and $\mathfrak{b} = \mathfrak{a} \cap \{\kappa^+ | \kappa \in S\}$ in case (b), we still have $\max \mathrm{pcf}\, \mathfrak{b} = \lambda^+$, because $\mathfrak{b}$ is unbounded in λ, hence $\max \mathrm{pcf}\, \mathfrak{b} \geq \lambda^+$, and because $\max \mathrm{pcf}\, \mathfrak{b} \leq \max \mathrm{pcf}\, \mathfrak{a} = \lambda^+$, since $\mathfrak{b} \subseteq \mathfrak{a}$.

Assume, without loss of generality, that $\lambda' > (\mathrm{cf}\, \lambda)^+$ in (a), and that $\inf S > (\mathrm{cf}\, \lambda)^+$ in (b). Since $|\mathfrak{b}| \leq |\mathfrak{a}| = \mathrm{cf}\, \lambda$, then $|\mathfrak{b}|^+ < \min \mathfrak{b}$, hence, by [She, II, Lemma 3.1], $\lambda^+ = \max \mathrm{pcf}\, \mathfrak{b} = \mathrm{cf} \prod_{\kappa \in \mathfrak{b}} \kappa$. Then Proposition 2 implies that F is either λ-decomposable, or λ^+-decomposable.

The last statements follow from Properties 3(a)-(e). □

Corollary 5. *If λ is a singular cardinal and the ultrafilter D is not* $\mathrm{cf}\, \lambda$*-decomposable, then the following conditions are equivalent:*

(a) There is $\lambda' < \lambda$ such that D is κ-decomposable for all regular cardinals κ with $\lambda' < \kappa < \lambda$.

(a′) (Only in case $\mathrm{cf}\, \lambda > \omega$*) $\{\kappa < \lambda | F^+$ is κ^+-decomposable$\}$ is stationary in λ.*

(b) D is λ^+-decomposable.

(c) There is $\lambda' < \lambda$ such that D is (λ', λ^+)-regular.

(d) D is (λ, λ)-regular.

(e) There is $\lambda' < \lambda$ such that D is (λ', λ)-regular.

(f) There is $\lambda' < \lambda$ such that D is (λ'', λ'')-regular for every λ'' with $\lambda' < \lambda'' < \lambda$.

Proof. (a) $\Rightarrow$ (b) and (a′) $\Rightarrow$ (b) are immediate from Theorem 4 and Property 3(a). In case $\mathrm{cf}\, \lambda > \omega$, (a) $\Rightarrow$ (a′) is trivial.

(b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (a) are given, respectively, by Properties 3(d)(c)(f)(c)(g). □

Corollary 6. *If λ is a singular cardinal, then an ultrafilter is (λ, λ)-regular if and only if it is either* $\mathrm{cf}\, \lambda$*-decomposable or λ^+-decomposable.*

Proof. Immediate from Corollary 5(d)$\Rightarrow$(b) and Properties 3(b)-(d). □

A topological space is $[\mu, \lambda]$-*compact* if and only if every open cover by λ many sets has a subcover by $< \mu$ many sets. A family $\mathcal{F}$ of topological spaces is *productively* $[\mu, \lambda]$-compact if and only if every (Tychonoff) product of members of $\mathcal{F}$ is $[\mu, \lambda]$-compact.

Corollary 7. *If λ is a singular cardinal, then a family of topological spaces is productively $[\lambda, \lambda]$-compact if and only if it is either productively* $[\mathrm{cf}\, \lambda, \mathrm{cf}\, \lambda]$*-compact or productively $[\lambda^+, \lambda^+]$-compact.*

Proof. Immediate from Corollary 6, Property 3(g) and [L3, Theorem 3] (see also [Ca]). □

Henceforth, by a *logic*, we mean a *regular logic* in the sense of [E]. Typical examples of regular logics are infinitary logics, or extensions of first-order logic obtained by adding new quantifiers; e. g., cardinality quantifiers asserting "there are at least ω_α x's such that …".

A logic L is $[\lambda, \mu]$-*compact* if and only if for every pair of sets Γ and Σ of sentences of L, if $|\Sigma| \leq \lambda$ and if $\Gamma \cup \Sigma'$ has a model for every $\Sigma' \subseteq \Sigma$ with $|\Sigma| < \mu$, then $\Gamma \cup \Sigma$ has a model (see [Ma] for some history and further comments).

Corollary 8. *If λ is a singular cardinal, then a logic is $[\lambda, \lambda]$-compact if and only if it is either $[\operatorname{cf} \lambda, \operatorname{cf} \lambda]$-compact or $[\lambda^+, \lambda^+]$-compact.*

Proof. Immediate from Corollary 6, Property 3(g) and [Ma, Theorem 1.4.4] (notice that in [Ma] in the definition of (λ, μ)-regularity for an ultrafilter the order of μ and λ is reversed). □

Theorem 9. *Suppose that $(\lambda_i)_{i \in I}$ and $(\mu_j)_{j \in J}$ are sets of infinite cardinals. Then the following are equivalent:*

(i) For every $i \in I$ there is a (λ_i, λ_i)-regular ultrafilter which for no $j \in J$ is (μ_j, μ_j)-regular.

(ii) There is a logic which is $[\lambda_i, \lambda_i]$-compact for every $i \in I$, and which for no $j \in J$ is $[\mu_j, \mu_j]$-compact.

(iii) For every $i \in I$ there is a $[\lambda_i, \lambda_i]$-compact logic which for no $j \in J$ is $[\mu_j, \mu_j]$-compact.

The logics in (ii) and (iii) can be chosen to be generated by at most $2 \cdot |J|$ cardinality quantifiers.

Proof. In the case when all the μ_j's are regular, the Theorem is proved in [L1, Theorem 4.1]. The general case follows from the above particular case, by applying Corollaries 6 and 8. □

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